Nonlinear differential equations with exact solutions expressed via the Weierstrass function

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Abstract

New problem is studied that is to find nonlinear differential equations with special solutions expressed via the Weierstrass function. Method is discussed to construct nonlinear ordinary differential equations with exact solutions. Main step of our method is the assumption that nonlinear differential equations have exact solutions which are general solution of the simplest integrable equation. We use the Weierstrass elliptic equation as a building block to find a number of nonlinear differential equations with exact solutions. Nonlinear differential equations of the second, third and fourth order with special solutions expressed via the Weierstrass function are given. Most of these equations are used at the description of nonlinear waves in physics.

Keywords: Nonlinear differential equation, exact solution, Weierstrass function, nonlinear evolution equation

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1 Introduction

Last years we can observe a splash of papers with methods of finding exact solutions of nonlinear differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9]. There are two reasons to make the study of this direction. First there is a great interest to the investigation of nonlinear processes. Secondly we have codes *Maple, Mathematica* and other ones to have a lot of symbolical calculations.

It is well known that all nonlinear differential equations can be conditionally divided into three types: exactly solvable, partially solvable and those that have no exact solution.

Usually investigators solve two problems in the theory of nonlinear differential equations. First problem is to find nonlinear solvable differential equations and to study their properties. The second problem is to search exact solutions of nonlinear differential equations.

This paper is devoted to solution of new problem that is to find nonlinear differential equations with special solutions. It is important to note that these equations have exact solutions but they are not all integrable equations. Using our approach we extend class of studied differential equations.

Consider nonlinear evolution equation

$$E_1[u] \equiv E_1(u, u_t, u_x, ..., x, t) = 0 \tag{1.1}$$

Assume we need to have exact solutions of this equation. Usually we look for exact solution of nonlinear evolution equation taking the travelling wave into account and search exact solution of equation (1.1) in the form

$$u(x,t) = y(z), \qquad z = x - C_0 t \tag{1.2}$$

As a result we have that the equation (1.1) reduces to the nonlinear ordinary differential equation (ODE)

$$E_2[y] \equiv E_2(y, y_z, ..., z) = 0 \tag{1.3}$$

To obtain exact solutions of equation (1.3) one can apply different approaches [10, 11, 12, 13, 14, 15, 16, 17]. However one can note that the most methods that are used to search exact solutions take the singular analysis into account for solutions of the nonlinear differential equations.

Using the singular analysis first of all one can consider the leading members of equation (1.3). After that one can find the singularity for solution of equation (1.3). Further the truncated expansion is used to have the transformation to search exact solutions of nonlinear ODEs. At this point one can use some trial functions (hyperbolic and elliptic and so on) to look for exact solutions of nonlinear ODEs.

However one can note that hyperbolic and elliptic functions are general solutions of nonlinear exactly solvable equations. We have as a rule that partially solvable nonlinear differential equations have exact solutions that are general solutions of solvable equations of lesser order.

This paper is the extension of our recent work [18] where we started the solution of the new problem that is to find nonlinear ordinary differential equations of polynomial form which have special solutions expressed via general solution of the Riccati equation.

Let us explain the idea of this work. It is well known there is the great problem that is to find nonlinear integrable differential equations. It is very important because we often want to have general solution of nonlinear ordinary equations. However sometimes we can content ourselves with some special solutions because a lot of differential equations are nonintegrable ones although they are intensively used in physics and look as simple equations. It is important to find some special solutions of these equations that are called exact solutions. In this paper we want to find nonlinear differential equations that are not all integrable but have special solutions in the form of the Weierstrass function.

The first aim of this work is to present the method to find nonlinear differential equations with exact solutions in the form of the Weierstrass function. The second aim of this paper is to give nonlinear ordinary differential equations that have exact solutions expressed via general solution of the Weierstrass elliptic equation.

The outline of this paper is as follows. In section 2 we present the method to find nonlinear ordinary differential equations (ODEs) with special solutions expressed via the Weierstrass function. Nonlinear ODEs with exact solutions of the first, second, third and fourth degree singularities are given in section 3, 4, 5 and 6. Example of nonlinear ODE with exact solution of the fifth degree singularity that is popular at the description of the model chaos is considered in section 7.

2 Method applied

Let us discuss the method that can be applied to find nonlinear differential equations with exact solutions expressed via the Weierstrass function. One can note that most nonlinear ordinary differential equations has exact solutions that are general solutions of differential equations of lesser order. Much more than that exact equations for the most nonlinear differential equations are determined via general solution of the Riccati equation. This is so indeed because one can note that the most approaches to search exact solutions of nonlinear ordinary differential equations are based on general solution of the Riccati equation. The application of the tanh method confirms this idea [18, 19, 20, 21]. We gave these nonlinear differential equations in our recent work [22].

However one can note that a lot of nonlinear ODEs have exact solutions that are general solutions of the Weierstrass elliptic function [1, 12, 14, 15, 23]. In this connection in this paper we are going to find nonlinear ODEs with exact solutions expressed via the Weierstrass function.

The Weierstrass elliptic equation can be presented in the form

$$P_1[R] = R_z^2 - 4R^3 + g_2 R + g_3 = 0 \tag{2.1}$$

We have the following simple theorem.

Theorem 2.1. *Let $R(z)$ be solution of equation* (2.1) *than equations*

$$R_{zz} = 6R^2 - \frac{1}{2}g_2 \tag{2.2}$$

$$R_{zzz} = 12RR_z \tag{2.3}$$

$$R_{zzzz} = 120R^3 - 18g_2R - 12g_3 \tag{2.4}$$

$$R_{zzzzz} = 360R^2R_z - 18g_2R_z \tag{2.5}$$

have special solutions expressed via general solution of equation (2.1).

Proof. Theorem 2.1 is proved by differentiation of (2.1) with respect to z and substitution R_z^2 from equation (2.1) and so on into expressions obtained.

□

It is well known that the general solution of equation (2.1) is the Weierstrass function

$$R(z) = \wp(z, g_2, g_3) \tag{2.6}$$

where g_2 and g_3 are arbitrary constants that are called invariants.

We want to find nonlinear ordinary differential equation which have special solutions that are determined via general solution of the Weierstrass elliptic equation.

Algorithm of our method can be presented by *four steps.* At *the first step* we choose the singularity of special solution and give the form of this solution. At *the second step* we take the order of nonlinear ordinary differential equation what we want to search. *The third step* lies in the fact that we write the general form of nonlinear differential equation taking the singularity of the solution into account and the given order for nonlinear differential equation. *The fourth step* contains calculations. As a result we find limitations for the parameters in order for nonlinear differential equation has exact solutions. At this step we can have nonlinear ODE with exact solutions.

3 Nonlinear ODEs with exact solutions of the first degree singularity

Let us demonstrate our approach to find nonlinear ODEs with exact solutions of the first degree singularity. These solutions can be presented in the form

$$y(z) = A_0 + A_1 \frac{R_z}{R} \tag{3.1}$$

Here $R(z)$ is a solution of the Weierstrass elliptic equation, A_0 and A_1 are unknown constants. First of all we are going to find nonlinear second order ODEs.

Second order ODEs. General form of the second order ODEs can be presented as the following

$$y_{zz} + a_1 y y_z + a_2 y^3 + b_0 y_z + b_1 y^2 - C_0 y + C_1 = 0 \tag{3.2}$$

Equation (3.2) was written taking singularity of solution (3.1) into account and the given order of nonlinear ODE what we want to have with exact solutions.

Let us assume $A_1 = 1$. Substituting $y(z)$ from (3.1) into (3.2) and using

(2.1) and (2.2) we have equation in the form

$$(2\,b_0 + 12\,a_2A_0 + 4\,b_1 + 2\,a_1A_0)\,R^4 +$$
$$+\left(4\,a_2R_z + 2\,R_z + a_2{A_0}^3 - C_0A_0 + 2\,a_1R_z + C_1 + b_1{A_0}^2\right)R^3 +$$
$$+\left(2\,b_1A_0R_z - 3\,a_2A_0g_2 - b_1g_2 + 3\,a_2{A_0}^2R_z - C_0R_z\right)R^2 + \tag{3.3}$$
$$+\left(b_0g_3 + a_1A_0g_3 - a_2R_zg_2 - 3\,a_2A_0g_3 - b_1g_3\right)R -$$
$$-a_2R_zg_3 - 2\,R_zg_3 + a_1R_zg_3 = 0$$

From the last equation we get relations for constants

$$a_1 = -1 - 2\,a_2, \quad C_0 = 3\,a_2{A_0}^2 + 2\,b_1A_0, \quad g_2 = 0 \tag{3.4}$$

$$a_2 = -1, \quad A_0 = \frac{1}{5}b_0 + \frac{2}{5}b_1 \tag{3.5}$$

$$C_1 = -\frac{7}{125}\,{b_0}^2b_1 - \frac{4}{125}\,b_0{b_1}^2 + \frac{4}{125}\,{b_1}^3 - \frac{2}{125}\,{b_0}^3, \quad b_1 = -3\,b_0 \tag{3.6}$$

Taking these constants into account we have equation in the form

$$y_{zz} + yy_z - y^3 + b_0y_z - 3\,b_0y^2 - 3\,{b_0}^2y - {b_0}^3 = 0 \tag{3.7}$$

Solution of this equation is expressed by the formula

$$y\,(z) = -b_0 + \frac{R_z}{R} \tag{3.8}$$

At $b_0 = 0$ from (3.7) we have equation that was found by P. Painleve [24].

Third order ODEs. The general case of the third order ODEs with exact solutions (3.1) takes the form

$$a_0y_{zzz} + a_1yy_{zz} + a_2{y_z}^2 + a_3y^4 + b_0y_{zz} + b_1yy_z + b_2y^3 +$$
$$+d_0y_z + d_1y^2 - C_0y + C_1 = 0 \tag{3.9}$$

Assuming $A_1 = 1$ without loss of the generality and substituting (3.1) into (3.9) we have coefficients in the form

$$b_2 = -\frac{1}{2}a_1A_0 - \frac{1}{2}b_1 - \frac{1}{2}b_0 - 4\,a_3A_0 \tag{3.10}$$

$$C_0 = -\frac{3}{2} b_1 {A_0}^2 + 2\, d_1 A_0 - 8\, a_3 {A_0}^3 - \frac{3}{2} {A_0}^3 a_1 - \frac{3}{2} {A_0}^2 b_0 \tag{3.11}$$

$$b_1 = 0, \quad b_0 = -a_1 A_0 \tag{3.12}$$

$$a_3 = -\frac{1}{4} a_2 - \frac{3}{4}\, a_0 - \frac{1}{2} a_1 \tag{3.13}$$

$$d_1 = -\frac{1}{2} d_0 - \frac{3}{2} {A_0}^2 a_2 - \frac{9}{2} {A_0}^2 a_0 - 3\, {A_0}^2 a_1 \tag{3.14}$$

$$\begin{gathered} C_1 = -4\, g_2 a_2 - 6\, a_0 g_2 - 2\, a_1 g_2 - \\ -\frac{1}{4} {A_0}^4 a_2 - \frac{3}{4} {A_0}^4 a_0 - \frac{1}{2} {A_0}^4 a_1 - \frac{1}{2} {A_0}^2 d_0 \end{gathered} \tag{3.15}$$

$$a_2 = -\frac{1}{6}\, \frac{-6\, a_1 g_3 + 18\, a_0 g_3 + g_2 d_0}{g_3}, \quad d_0 = \frac{a_0 g_2^2}{g_3} \tag{3.16}$$

$$a_1 = \frac{a_0 \left({g_2}^3 + 108\, {g_3}^2 \right)}{24 g_3^2} \tag{3.17}$$

We also have two relations for constants g_3

$$g_3 = \pm \frac{1}{6}\, {g_2}^{3/2} \tag{3.18}$$

As a result we have the following equations

$$\begin{gathered} y_{zzz} + (6\, y - 6\, A_0)\, y_{zz} - 3 {y_z}^2 - 3\, y^4 + 12\, A_0 y^3 + \\ + \left(-18\, {A_0}^2 \mp 3\, \sqrt{g_2} \right) y^2 + \left(\pm 6\, A_0 \sqrt{g_2} + 12\, {A_0}^3 \right) y \mp \\ \mp 3\, {A_0}^2 \sqrt{g_2} - 3\, {A_0}^4 \pm 6\, \sqrt{g_2} y_z + 6\, g_2 = 0 \end{gathered} \tag{3.19}$$

with exact solutions

$$y\,(z) = A_0 + \frac{R_z}{R} \tag{3.20}$$

Fourth order ODEs. Now let us find nonlinear fourth order ODEs with exact solutions of the first degree singularity. We have the following nonlinear fourth order ODE of the general form

$$a_0 y_{zzzz} + a_1 y y_{zzz} + a_2 y^2 y_{zz} + a_3 y^3 y_z + a_4 y^5 + a_5 y {y_z}^2 + a_6 y_z y_{zz} +$$

$$+ b_0 y_{zzz} + b_1 y y_{zz} + b_2 {y_z}^2 + b_3 y^4 + d_0 y_z + d_1 y y_z + d_2 y^3 + \tag{3.21}$$

$$+ h_0 y_z + h_1 y^2 - C_0 y + C_1 = 0$$

Assuming $g_2 = m^2$ and substituting (3.1) at $A_0 = 0$ and $A_1 = 1$ into (3.21) we have coefficients in the form

$$a_6 = -6\,a_0 - 3\,a_1 - 4\,a_4 - 2\,a_2 - 2\,a_3 - a_5 \tag{3.22}$$

$$d_2 = -\frac{1}{2} d_1 - \frac{1}{2} d_0 \tag{3.23}$$

$$C_0 = -4\,a_2 m^2 - 8\,a_4 m^2 + 2\,a_5 m^2 \tag{3.24}$$

$$a_5 = a_2 - a_3 - 3\,a_1 - \frac{1}{6}\frac{m^2 d_1}{g_3}, \tag{3.25}$$

$$a_4 = -\frac{3}{2}\,a_0 + \frac{3}{4}\,a_1 - \frac{3}{4}\,a_2 + \frac{1}{4}\,a_3 + \frac{3}{4}\,\frac{d_0 g_3}{m^4} - \frac{3}{4}\,\frac{d_1 g_3}{m^4} + \frac{1}{24}\,\frac{d_1 m^2}{g_3} \tag{3.26}$$

$$a_3 = -6\,\frac{d_0 g_3}{m^4} + \frac{1}{12}\,\frac{d_1 m^2}{g_3} + 3\,a_1 - 15\,a_0 + 6\,\frac{d_1 g_3}{m^4} - 2\,a_2 \tag{3.27}$$

$$a_2 = -5\,a_0 + 2\,a_1 + \frac{1}{36}\,\frac{d_1 m^2}{g_3} - 3\,\frac{d_0 g_3}{m^4} + 3\,\frac{d_1 g_3}{m^4} \tag{3.28}$$

$$b_3 = -\frac{3}{4} b_0 - \frac{1}{4} b_2 - \frac{1}{2} b_1, \quad h_1 = -\frac{1}{2} h_0 \tag{3.29}$$

$$C_1 = -6\,b_0 m^2 - 4\,b_2 m^2 \tag{3.30}$$

$$b_2 = b_1 - 3\,b_0 - \frac{1}{6}\frac{h_0 m^2}{g_3} \tag{3.31}$$

$$h_0 = \frac{b_0 m^4}{g_3} \tag{3.32}$$

$$b_0 = \frac{1}{16}\frac{h_0 g_3}{m^4} + \frac{1}{36}\frac{h_0 m^2}{g_3} \tag{3.33}$$

$$g_3^{(1,2)} = \pm\frac{1}{6}m^3 \tag{3.34}$$

As a result we have equation in the form at $g_3 = \frac{1}{6}m^3$

$$\begin{gathered}
a_0 y_{zzzz} + \frac{1}{6}\,(6\,a_1 y + b_1)\,y_{zzz} + \left(10\,a_0 - a_1 + \frac{1}{2}\frac{d_0}{m} - \frac{1}{6}\frac{d_1}{m}\right) y_z y_{zz} + \\
+ (d_0 + b_1 y)\,y_{zz} + \left(2\,a_1 - \frac{2}{3}\frac{d_1}{m} + \frac{1}{2}\frac{d_0}{m} - 5\,a_0\right) y^2 y_{zz} + \\
+ \left(\frac{1}{2}\frac{d_1}{m} + \frac{1}{2}\frac{d_0}{m} + b_2\right) y {y_z}^2 + \left(a_0 - \frac{1}{2}\frac{d_0}{m} - a_1 + \frac{1}{3}\frac{d_1}{m}\right) y^5 - \\
- \left(\frac{1}{4}\,b_2 + \frac{5}{8}\,b_1\right) y^4 - \left(\frac{1}{2}\,d_1 + \frac{1}{2}\,d_0\right) y^3 + \frac{1}{2}\,b_1 m y^2 + \\
+ \left(d_1 y - 5\,y^3 a_0 - y^3 a_1 - \frac{1}{6}\frac{y^3 d_1}{m} - b_1 m\right) y_z - \\
- \left(m d_1 + 12\,m^2 a_0 - 3\,m d_0\right) y - 4\,b_2 m^2 - m^2 b_1 = 0
\end{gathered} \tag{3.35}$$

with exact solution in the form

$$y\,(z) = \frac{R_z}{R} \tag{3.36}$$

Where $R(z)$ is the solution of equation in the form

$$R_z^2 - 4R^3 + m^2 R + \frac{1}{6}m^3 = 0 \tag{3.37}$$

Solution of the equation (3.35) is expressed via the Weierstrass function and has the first degree singularity. At the present we do not know any application of equation (3.35).

4 Nonlinear ODEs with exact solutions of the second degree singularity

Let us find nonlinear ODEs with exact solutions of the second degree singularity expressed via the Weierstrass function

$$y(z) = A_0 + A_1 \frac{R_z}{R} + A_2 R \tag{4.1}$$

We do not consider the second order nonlinear ODEs with exact solutions (4.1) because this equation is the exactly solvable equation. We start to find the third order nonlinear ODEs with exact solutions of the second degree singularity expressed via the Weierstrass function.

Third order ODEs. Let us write the general form of the nonlinear third ODEs with solution of the second order singularity. It takes form

$$a_0 y_{zzz} + a_1 y y_z + b_0 y_{zz} + b_1 y^2 + d_0 y_z - C_0 y + C_1 = 0 \tag{4.2}$$

Substituting (4.1) into equation (4.2) we obtain the following parameters

$$a_1 = -12\, a_0, \quad d_0 = 12\, a_0 A_0, \quad b_1 = \frac{C_0}{2A_0} \tag{4.3}$$

$$A_1 = 0, \quad A_0 = -\frac{C_0}{12 b_0} \tag{4.4}$$

$$g_2 = \frac{{C_0}^2 + 24 C_1 b_0}{12 b_0^2} \tag{4.5}$$

As a result we have equation in the form

$$a_0 y_{zzz} - 12 a_0 y y_z + b_0 y_{zz} - \frac{a_0 C_0}{b_0} y_z - 6 b_0 y^2 - C_0 y + C_1 = 0 \tag{4.6}$$

with exact solutions

$$y\,(z) = -\frac{C_0}{12 b_0} + R\,(z) \tag{4.7}$$

Equation (4.6) can be found from the enough popular nonlinear evolution equation

$$u_t + \lambda_1 u u_x + \lambda_2 (u u_x)_x + \lambda_3 u_{xx} + \lambda_4 u_{xxx} + \lambda_5 u_{xxxx} = 0 \tag{4.8}$$

Nonlinear evolution equation (4.8) was used at the description of nonlinear wave processes and studied in works [25, 26, 27, 28]. Exact solutions of this equation were obtained in works [29, 30]. They also rediscovered later.

Fourth order ODEs. Now let us find the nonlinear fourth order ODEs with exact solutions of the second degree singularity expressed via the Weierstrass function. The general case of this nonlinear ODE can be presented in the form

$$a_0 y_{zzzz} + a_1 y y_{zz} + a_2 {y_z}^2 + a_3 y^3 + b_0 y_{zzz} + b_1 y y_z + \\ + d_0 y_{zz} + d_1 y^2 + h_0 y_z - g_2 C_0 y + C_1 = 0 \tag{4.9}$$

We assume that exact solutions of equation (4.9) can be found by the formula (4.1).

Without loss of generality let us assume in (4.1) $A_0 = 0$ and $A_2 = 1$. We obtain the following values parameters for equation (4.9)

$$b_1 = -12\, b_0, \quad h_0 = 0, \quad A_1 = 0 \tag{4.10}$$

$$a_3 = -120\, a_0 - 6\, a_1 - 4\, a_2, \quad d_1 = -6\, d_0 \tag{4.11}$$

$$a_2 = -18 a_0 - \frac{1}{2} a_1 - C_0 \tag{4.12}$$

$$C_1 = \frac{1}{2}\, d_0 g_2 - \frac{1}{2}\, g_3 a_1 - 6\, a_0 g_3 - g_3 C_0 \tag{4.13}$$

We get nonlinear ODEs in the form

$$a_0 y_{zzzz} + b_0 y_{zzz} + (a_1 y + d_0)\, y_{zz} - \left(18\, a_0 + \frac{1}{2}\, a_1 + C_0\right) {y_z}^2 - \\ -12\, b_0 y y_z + (4\, C_0 - 48\, a_0 - 4\, a_1)\, y^3 - 6\, d_0 y^2 - g_2 C_0 y + \\ + \frac{1}{2}\, d_0 g_2 - \frac{1}{2}\, g_3 a_1 - 6\, a_0 g_3 - g_3 C_0 = 0 \tag{4.14}$$

Exact solutions of equation (4.14) is found by the formula

$$y\,(z) = R\,(z) \tag{4.15}$$

Assuming $a_0 = 1$, $b_0 = 0$, $d_0 = 0$ in (4.14) we have equation

$$y_{zzzz} + a_1 y y_{zz} - \left(18 + \frac{1}{2}\, a_1 + C_0\right) {y_z}^2 + \\ +4\,(C_0 - 12 - a_1)\, y^3 - g_2 C_0 y - 6\, g_3 - \frac{1}{2}\, g_3 a_1 - g_3 C_0 = 0 \tag{4.16}$$

One can note that the last equation has two parameters in leading members. However let us show that this equation contains a lot of important nonlinear integrable and nonintegrable differential equations.

Assuming $a_1 = -30$, $C_0 = -3$ and $y(z) \to -y(z)$ in equation (4.16) we have equation in the form

$$y_{zzzz} + 30\,y y_{zz} + 60\,y^3 + 3\,g_2 y - 12\,g_3 = 0 \tag{4.17}$$

Equation (4.17) is exactly solvable equation. This equation can be obtained by the travelling wave (1.2) from the Caudrey – Dodd – Gibbon equation [31, 32, 34]

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 30 u u_{xx} + 60 u^3\right) = 0 \tag{4.18}$$

Equation (4.18) is integrable equation by the inverse scattering transform. However we can see that this equation is found in our class of nonlinear differential equations.

Assuming $a_1 = -20$, $C_0 = 2$ and $y(z) \to -y(z)$ in equation (4.16) we have equation in the form

$$y_{zzzz} + 20\,y y_{zz} + 10\,{y_z}^2 + 40\,y^3 - 2\,g_2 y - 2\,g_3 = 0 \tag{4.19}$$

Using the travelling wave we can see that equation (4.19) is obtained from the Korteveg - de Vries equation of the fifth order [32, 34]

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 20 u u_{xx} + 10 u_x^2 + 40 u^3\right) = 0 \tag{4.20}$$

We have obtained that special solutions of the fifth order Korteveg - de Vries equation can be found by the formula (4.15).

Assuming $a_1 = -15$, $C_0 = \frac{3}{4}$ and $y(z) \to -\frac{2}{3}y(z)$ in (4.16) we have equation with exact solution (4.15) in the form

$$y_{zzzz} + 10\,y y_{zz} + \frac{15}{2} y_z^2 + \frac{20}{3} y^3 - \frac{3}{4}\,g_2 y - \frac{9}{8}\,g_3 = 0 \tag{4.21}$$

Equation (4.21) is exactly solvable equation too. Using the travelling wave this one can be obtained from the Kaup - Kupershmidt equation [32, 33, 34]

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 10 u u_{xx} + \frac{15}{2} u_x^2 + \frac{20}{3} u^3\right) = 0 \tag{4.22}$$

Assuming $a_1 = -15$, $C_0 = \frac{3}{4}$ and $y(z) \to -y(z)$ in (4.16) we have equation

$$y_{zzzz} - 15\,y y_{zz} + \frac{45}{4}\,{y_z}^2 + 15\,y^3 - \frac{3}{4}\,g_2 y - \frac{3}{4}\,g_3 = 0 \tag{4.23}$$

Equation (4.23) is also exactly solvable equation [35]. This one has exact solution (4.15) as well. This equation can be found from the Schwarz – Kaup–Kuperschmidt equation of the fifth order, that is the singular manifold equation for the Kaup – Kuperschmidt equation. The Cauchy problems for these equations can be solved by inverse scattering transform.

Assuming $a_1 = -12$, $C_0 = 0$ and $y(z) \to -y(z)$ in (4.16) we have equation

$$y_{zzzz} + 12\, y y_{zz} + 12\, {y_z}^2 = 0 \tag{4.24}$$

Equation (4.24) is exactly solvable equation again [35]. The general solution of this equation is expressed via the first Painleve transcendent. However this equation has special solution expressed by the formula (4.15) as well.

We have interesting result that is a number of exactly solvable equations are found in our class of differential equations. Much more than that these nonlinear ODEs have the similar special solutions expressed by the formula (4.15) via the Weierstrass function. We are going to use this observation in future work to look for exactly solvable nonlinear ODEs of higher order.

Assuming $a_1 = 0$ and $C_0 = -18a_0$ in (4.14) we have equation

$$\begin{gathered} a_0 y_{zzzz} + b_0 y_{zzz} + d_0 y_{zz} - 12\, b_0 y y_z - 120\, y^3 a_0 - \\ -6\, d_0 y^2 + 18\, g_2 a_0 y + C_1 = 0 \end{gathered} \tag{4.25}$$

At $b_0 = 0$ we have nonlinear ODEs

$$a_0 y_{zzzz} + d_0 y_{zz} - 120 a_0\, y^3 - 6\, d_0 y^2 + 48\, g_2 a_0 y + C_1 = 0 \tag{4.26}$$

Last equation corresponds to the nonlinear evolution equation that is used at the description of the nonlinear dispersive waves [36, 37, 38, 39]

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} + \delta u_{xxxxx} = 0 \tag{4.27}$$

We have equation (4.26) if we look for exact solution of equation (4.27) in the form of the travelling wave (1.2).

In the case $d_0 = 0$ from (4.25) we obtain the nonlinear ODEs in the form

$$a_0 y_{zzzz} - 120 a_0\, y^3 + 48 a_0\, g_2 y + C_1 = 0 \tag{4.28}$$

Equation (4.28) are used at the description of nonlinear dispersive waves as well. Corresponding nonlinear evolution equation can be met at the the description of the nonlinear waves and takes the form [40, 41]

$$u_t + \beta u^2 u_x + \delta u_{xxxxx} = 0 \tag{4.29}$$

Exact solutions of equations (4.25),(4.26) and (4.28) are found by the formula (4.15).

5 Nonlinear ODEs with the third degree singularity solution

In this section we are going to find nonlinear ODEs that have exact solutions of the third degree singularity and are expressed via the Weierstrass function. These solutions can be presented by the formula

$$y\left(z\right)=A_0+A_1\frac{R_z}{R}+A_2R+A_3R_z \tag{5.1}$$

We can not suggest any nonlinear ODEs of the second order with exact solutions (5.1) because we can not have the polynomial class of this equation. Let us start to consider the third nonlinear ODE.

Third order ODE. We have the general case of the nonlinear third order ODE in the form

$$a_0y_{zzz}+a_1y^2+b_0y_{zz}+d_0y_z-C_0y+C_1=0 \tag{5.2}$$

Assuming $A_3=1$ and $g_2=\frac{n^2}{12}$ we have following relations for the parameters

$$d_0=-2a_1A_1+6\frac{b_0A_1}{A_2}+\frac{C_0}{A_2}-2\,\frac{a_1A_0}{A_2} \tag{5.3}$$

$$A_1=0,\quad b_0=-a_0A_2-\frac{1}{6}a_1A_2 \tag{5.4}$$

$$a_1=-30\,a_0,\quad C_0=-a_0\left(60\,A_0-{A_2}^3\right) \tag{5.5}$$

We obtain four values for constant A_2

$$A_2^{(1,2)}=\pm n,\quad A_2^{(3,4)}=\pm in \tag{5.6}$$

and four values for constant C_1

$$C_1^{(1,2)}=-\frac{a_0}{24}\left(432g_3+720A_0^2\mp 24A_0n^3-5n^6\right) \tag{5.7}$$

$$C_1^{(3,4)}=i\frac{a_0}{24}\left(432ig_3+720iA_0^2\mp 24A_0n^3+5in^6\right) \tag{5.8}$$

We have four nonlinear ODEs

$$a_0y_{zzz}-30a_0y^2\pm 4a_0ny_{zz}+a_0n^2y_z\pm a_0n^3y+60a_0A_0y+C_1=0 \tag{5.9}$$

$$a_0y_{zzz}-30a_0y^2\pm 4ia_0ny_{zz}-a_0n^2y_z\mp ia_0n^3y+60a_0A_0y+C_1=0 \tag{5.10}$$

with exact solutions

$$y\,(z) = A_0 \pm nR + R_z \tag{5.11}$$

$$y\,(z) = A_0 \pm inR + R_z \tag{5.12}$$

Equation (5.9) can be found from nonlinear evolution equation

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} + \delta u_{xxxx} = 0 \tag{5.13}$$

This is the Kuramoto – Sivashinsky equation [42, 43, 44]. Equation (5.13) is used at the description of turbulence processes [44, 45, 46]. Solution of this equation was found in works [10, 12, 14, 15].

From equation (5.9) and (5.10) one can see there is the special solution of equation (5.13) at $\beta = 0$ and $\gamma = 0$. This equation reminds the Burgers equation and takes the form

$$u_t + \alpha u u_x + \delta u_{xxxx} = 0 \tag{5.14}$$

Exact solutions of equation (5.14) is found by the formula

$$y\,(z) = A_0 + R_z \tag{5.15}$$

if we look for solution of equation (5.14) in the form of the travelling wave. This equation takes the form

$$\delta y_{zzzz} + \frac{\alpha}{2} y^2 - C_0 y + \frac{C_0^2 + 2160\delta^2 g_3}{2\alpha} = 0 \tag{5.16}$$

There is the rational solution of equation (5.16) in the form

$$y\,(z) = \frac{C_0}{\alpha} + \frac{120\delta}{\alpha(z - z_0)^3} \tag{5.17}$$

In this case equation (5.16) can be written in the simple form

$$\delta y_{zzzz} + \frac{\alpha}{2} y^2 - C_0 y + \frac{C_0^2}{2\alpha} = 0 \tag{5.18}$$

We found nonlinear ODEs that correspond to the Kuramoto - Sivashinsky equation. These equations are very popular at the description of the turbulence processes.

Fourth order ODEs. Let us find the nonlinear ODEs of fourth order with exact solutions (5.1). We can use the following general form of the fourth order

$$a_0 y_{zzzz} + a_1 y y_z + b_0 y_{zzz} + b_1 y^2 + d_0 y_{zz} + h_0 y_z - g_2 C_0 y + C_1 = 0 \tag{5.19}$$

Assuming $A_2 = -60$ and substituting (5.1) at $g_2 = 6A_2^4$ into equation (5.19) we obtain at $A_1 = 0$ relations for the parameters

$$a_1 = a_0 \tag{5.20}$$

$$d_0 = -\frac{1}{6}b_1 A_2 + \frac{1}{60}\, b_0 A_2 + \frac{1}{720}\, a_0 {A_2}^2 \tag{5.21}$$

$$h_0 = 4320\, a_0 {A_2}^3 - 360\, {A_2}^3 C_0 \tag{5.22}$$

$$b_1 = \frac{1}{2}b_0 + \frac{1}{30}a_0 A_2 \tag{5.23}$$

$$b_0 = \frac{1}{12}\, A_2\, (186623999\, a_0 - 15552000\, C_0) \tag{5.24}$$

$$C_0 = 12a_0 \tag{5.25}$$

$$C_1 = -18\, a_0 A_2 g_3 + \frac{1}{240}\, {A_2}^7 a_0 \tag{5.26}$$

Taking these values for the parameters into account we have nonlinear ODE in the form

$$\begin{gathered} a_0 y_{zzzz} - \frac{1}{12}\, A_2 a_0 y_{zzz} + \frac{1}{720}\, a_0 {A_2}^2 y_{zz} + a_0 y y_z - \frac{1}{120}\, a_0 A_2 y^2 - \\ -72\, {A_2}^4 a_0 y - 18\, a_0 A_2 g_3 + \frac{1}{240}\, {A_2}^7 a_0 = 0 \end{gathered} \tag{5.27}$$

Equation (5.27) can be found from the nonlinear evolution equation

$$u_t + \alpha (u u_x)_x + \beta u u_x + \gamma u_{xx} + \delta u_{xxx} + \varepsilon u_{xxxx} + u_{xxxxx} = 0 \tag{5.28}$$

if we look for exact solution in the form of the travelling wave (1.2).

Solution of this equation is expressed by the formula

$$y\,(z) = A_2 R - 60\, R_z \tag{5.29}$$

At $A_2 = 0$ from (5.27) we obtain the nonlinear ODE that corresponds to the partial case (5.14) of the Kuramoto - Sivashinsky equation again.

6 Nonlinear ODEs with exact solutions of the fourth order singularity

Let us find nonlinear ordinary differential equations which have exact solutions of the fourth order singularity. These solutions can be presented by the formula

$$y\left(z\right)=A_0+\frac{A_1R_z}{R}+A_2R+A_3R_z+A_4R^2 \tag{6.1}$$

where $R(z)$ satisfies the Weierstrass elliptic equation again.

We can not suggest nonlinear ODEs of the second and third order of the polynomial form with solution (6.1). However we can consider the nonlinear fourth order ODE that takes the form.

$$a_0y_{zzzz}+a_1y^2+d_0y_{zz}+h_0y_z-g_2C_0y+C_1=0 \tag{6.2}$$

For calculations it is convenient to use $A_1=0,\ \ A_3=0$ and $A_4=-180$.

$$y\left(z\right)=A_0+A_2R-180\,R^2 \tag{6.3}$$

Substituting (6.1) into equation (6.2) we have

$$h_0=0,\quad a_1=\frac{14}{3}\,a_0,\quad d_0=-\frac{13}{30}\,a_0A_2 \tag{6.4}$$

$$A_0=\frac{3}{28}\,\frac{g_2C_0}{a_0}+\frac{31}{25200}\,{A_2}^2+18\,g_2 \tag{6.5}$$

$$g_3=-\frac{1}{58320000}\,A_2\left(31\,{A_2}^2-226800\,g_2\right) \tag{6.6}$$

$$\begin{gathered}C_1=-\frac{127}{300}\,a_0{A_2}^2g_2-\frac{961}{136080000}\,a_0{A_2}^4+\frac{3}{56}\,\frac{{g_2}^2{C_0}^2}{a_0}-\\-1242\,a_0{g_2}^2+168\,a_0A_2g_3\end{gathered} \tag{6.7}$$

In this case we get equation

$$\begin{gathered}a_0y_{zzzz}+\frac{14}{3}\,a_0y^2-\frac{13}{30}\,a_0A_2y_{zz}-g_2C_0y-\frac{127}{300}\,a_0{A_2}^2g_2-\\-\frac{961}{136080000}\,a_0{A_2}^4+\frac{3}{56}\,\frac{{g_2}^2{C_0}^2}{a_0}-1242\,a_0{g_2}^2+168\,a_0A_2g_3=0\end{gathered} \tag{6.8}$$

with exact solution

$$y\,(z) = A_0 + A_2 R - 180\,R^2 \tag{6.9}$$

Equation (6.8) corresponds to the nonlinear evolution equation [36, 37, 41]

$$u_t + \alpha u u_x + \beta u_{xxx} = \delta u_{xxxxx} \tag{6.10}$$

Exact solutions of equation (6.8) were found before in works [10, 13] and rediscovered in a number of papers later.

At $A_2 = 0$ from (6.8) we have nonlinear ODE in the form

$$a_0 y_{zzzz} + \frac{14}{3}\,a_0 y^2 - g_2 C_0 y - 1242\,a_0 {g_2}^2 + \frac{3}{56}\,\frac{{g_2}^2 {C_0}^2}{a_0} = 0 \tag{6.11}$$

This equation can be found from the nonlinear evolution equation [37, 41]

$$u_t + \alpha\, u u_x + \delta\, u_{xxxxx} = 0 \tag{6.12}$$

if we look for exact solutions of this nonlinear evolution equation in the form of the travelling wave. Special solution of equations (6.11) and (6.12) are found by the formula (6.9) at $A_2 = 0$.

7 Nonlinear ODE with exact solution of the fifth degree singularity

We have not got any possibility to suggest any nonlinear differential equation of the polynomial class of the second, third and fourth order with exact solution of the fifth degree singularity. We can look for such type of nonlinear fifth order ODE.

Assume we want to have nonlinear ODE with exact solution of the fifth degree singularity

$$y\,(z) = A_0 + \frac{A_1 R_z}{R} + A_2 R + A_3 R_z + A_4 R^2 + A_5 R R_z \tag{7.1}$$

Simplest case of the nonlinearity for this solution takes the form y^2 and we can see that (7.1) give singularity of the tenth degree. General case of equation takes the form

$$\varepsilon\, y_{zzzzz} + \gamma\, y_{zzzz} + \delta\, y_{zzz} + \chi\, y_{zz} + \beta\, y_z + \alpha\, y^2 - C_0 y + C_1 = 0 \tag{7.2}$$

This equation can be found from the nonlinear evolution equation [18, 47]

$$u_t + 2\alpha\, u u_x + \beta\, u_{xx} + \chi\, u_{xxx} + \delta\, u_{xxxx} + \gamma\, u_{xxxxx} + \varepsilon u_{xxxxxx} = 0 \tag{7.3}$$

if we search exact solutions in the form of the travelling wave (1.2).

Consider the simplest case of equation (7.3) taking $\gamma = 0$, $\chi = 0$ and $\alpha = \frac{1}{2}$ into account. In this case from (7.2) we get equation

$$\varepsilon\, y_{zzzzz} + \delta\, y_{zzz} + \beta\, y_z + \frac{1}{2}\, y^2 - C_0 y + C_1 = 0 \tag{7.4}$$

Let us find exact solutions of the nonlinear ODE (7.4). At $A_1 = 0$, $A_2 = 0$, $A_4 = 0$ we have relations for constants

$$A_5 = -15120\, \varepsilon, \quad A_3 = -\frac{1260}{11}\, \delta \tag{7.5}$$

$$\beta = \frac{10}{121}\, \frac{\delta^2}{\varepsilon} \tag{7.6}$$

$$g_3 = -\frac{7}{660}\, \frac{\delta\, g_2}{\varepsilon} - \frac{1}{574992}\, \frac{\delta^3}{\varepsilon^3}, \quad g_2 = \frac{1}{1452}\, \frac{\delta^2}{\varepsilon^2} + \frac{1}{5082}\, \frac{\delta^2 \sqrt{21}}{\varepsilon^2} \tag{7.7}$$

$$C_1 = -\frac{10854}{161051}\, \frac{\delta^5}{\varepsilon^3} - \frac{2484}{161051}\, \frac{\delta^5 \sqrt{21}}{\varepsilon^3} + \frac{1}{2} {C_0}^2 \tag{7.8}$$

Substituting (7.6) and (7.8) into equation (7.2) we obtain nonlinear ODE in the form

$$\begin{gathered}\varepsilon\, y_{zzzzz} + \delta\, y_{zzz} + \frac{10}{121}\, \frac{\delta^2 y_z}{\varepsilon} + \frac{1}{2} y^2 - C_0 y + \\ + \frac{1}{2} {C_0}^2 - \frac{10854}{161051}\, \frac{\delta^5}{\varepsilon^3} - \frac{2484}{161051}\, \frac{\delta^5 \sqrt{21}}{\varepsilon^3} = 0\end{gathered} \tag{7.9}$$

with exact solution

$$y\,(z) = C_0 - \frac{1260}{11} R_z\,(\delta + 132 \varepsilon R) \tag{7.10}$$

Where $R(z)$ satisfies the Weierstrass elliptic equation

$$R_z^2 - 4\, R^3 + \left(\frac{1}{1452}\, \frac{\delta^2}{\epsilon^2} + \frac{1}{5082}\, \frac{\delta^2 \sqrt{21}}{\epsilon^2} \right) R - \frac{7}{660}\, \frac{\delta\, g_2}{\epsilon} - \frac{1}{574992}\, \frac{\delta^3}{\epsilon^3} = 0 \tag{7.11}$$

Equation (7.9) is found from the nonlinear evolution equation

$$u_t + u u_x + \beta u_{xx} + \delta u_{xxxx} + \varepsilon u_{xxxxxx} = 0 \tag{7.12}$$

In the case $\delta = 0$ we obtain $\beta = 0$ and have the rational solution of equation (7.12). This solution takes the form

$$y(z) = C_0 + \frac{30240\varepsilon}{(z+z_0)^5} \tag{7.13}$$

Equation (7.9) in this case takes the form

$$\varepsilon\, y_{zzzzz} + \frac{1}{2}y^2 - C_0 y + \frac{1}{2}{C_0}^2 = 0 \tag{7.14}$$

Last years equation (7.12) is used at description of extensive model model chaos [48,49,50]. We hope that exact solution (7.10) will be useful at the study of the turbulence processes where equation (7.12) is applied.

Rational solutions (5.17) and (7.13) of equations (5.18) and (7.14) suggest us how one can find rational solutions of the generalized differential equations

$$u_t + \alpha u u_x + \varepsilon u_{n+1,x} = 0, \quad u_{n+1,x} = \frac{\partial^{(n+1)} u}{\partial x^{(n+1)}} \tag{7.15}$$

Using the travelling wave (2.1) we have from (7.15) the nonlinear ODE in the form

$$\varepsilon\, y_{n,z} + \frac{\alpha}{2}y^2 - C_0 y + \frac{1}{2}{C_0}^2 = 0, \quad y_{n,x} = \frac{d^n y}{dx^n} \tag{7.16}$$

Rational solutions of (7.16) takes the form

$$y(z) = \frac{C_0}{\alpha} + \frac{2\,(-1)^{(n-1)}\,(2n-1)!\,\varepsilon}{\alpha\,(n-1)!\,(z+z_0)^n} \tag{7.17}$$

Here z_0 is a arbitrary constant. Exact solution (7.17) can be useful at numerical simulation of the nonlinear problems where equations (7.15) and (7.16) are used.

8 Conclusion

Let us emphasize in brief the results of this work. We noted that a lot of nonlinear differential equations have exact solutions expressed via the Weierstrass function.

This observation provided the idea to find nonlinear differential equations with special solutions expressed via the Weierstrass function. We formulated new problem and found the polynomial class of nonlinear differential equations of the second, third and fourth order which have exact solutions expressed via the Weierstrass function. Exact solutions of these equations have different singularities and are expressed via general solutions of the Weierstrass elliptic equation.

We also list a number of nonlinear ODEs with exact solutions. These equations are found from the widely used nonlinear evolution equations taking the travelling wave into account.

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